APPLYING INEQUALITIES TO CALCULATING LIMITS

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Abstract: *in this article considered the use of inequalities to find the limits of fairly complex sequences. To this end, proved a few inequalities and with the help of these inequalities calculate limits. The fact is that the values determined from one practical task, or another can be found not exactly, but approximately. In the decision of practical problems have to take into account all error measurements. Moreover, as the improvement of the art and complications problems have to improve and technique measurement values. In the following tasks, using inequalities, we calculate the limits of fairly complex sequences.*
Keywords: *the limits of, sequence, inequality harmonic series, convergent, divergent*

ПРИМЕНЕНИЕ НЕРАВЕНСТВ ДЛЯ РАСЧЕТА ЛИМИТОВ

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Аннотация: *в этой статье рассматривается использование неравенства, чтобы найти достаточно сложные последовательности с этой целью доказано несколько неравенства и с помощью этих неравенств вычислены пределы. Определенные из того или иного практического задания, можно найти не точно, а приблизительно. При решении практических задач необходимо учитывать все погрешности измерений. Более того, по мере усовершенствования техники и проблем усложнения приходится совершенствовать и технику измерения*

значений. В следующих задачах, используя неравенства, мы вычисляем пределы довольно сложных последовательностей.
Ключевые слова: *пределы, последовательности неравенство, гармонический ряд, сходящийся, расходящийся.*

The important role of inequalities is determined by their application in various questions of natural scenic and technology.

The fact is that the values determined from one practical task, or another can be found not exactly, but approximately. In the decision of practical problems have to take into account all error measurements. Moreover, as the improvement of the art and complications problems have to improve and technique measurement values.

In the following tasks, using inequalities, we calculate the limits of fairly complex sequences.

Lemma. For any natural n, the following inequality holds

$$\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n} \qquad (1)$$

Evidence. It is known that a monotonically increasing and bounded variable has a limit. Therefore, there is a limit to the variable x_n. This limit is denoted by the letter e, i.e. [1]

$$e = \lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Since the value of x_n approaches its limit increasing, then x_n is less than its limit, i.e.

$$x_n = \left(1+\frac{1}{n}\right)^n < e \qquad (2)$$

It is easy to check that $e<3$. We now show that the limit of the variable Y_n is equal to e . In fact

$$\lim Y_n = \lim\left(1+\frac{1}{n}\right)^{n+1} = \lim\left(1+\frac{1}{n}\right)^n \cdot \left(1+\frac{1}{n}\right) = e\cdot 1 = e$$

As Y_n approaches number e decreasing, that

$$\left(1+\frac{1}{n}\right)^{n+1} > e \qquad (3)$$

Combining inequalities (2) and (3) we get

$$\left(1+\frac{1}{n}\right)^n > e < \left(1+\frac{1}{n}\right)^{n+1}$$

Logarithm these inequality in the basis of e, finally we find [2]

$$n\ln\left(1+\frac{1}{n}\right) < \ln e = 1 < (n+1)\ln\left(1+\frac{1}{n}\right), \quad \frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}$$

Lemma proved.

The task of 1. Believing

$x_1 = 1 + \frac{1}{2}, \quad x_2 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \quad x_3 = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6},$

$x_4 = 1 + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}, \quad x_n = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n},$

find $\lim\limits_{n\to\infty} x_n$.

The solution. Replacing n by $n-1$ in the first part of inequality (1), we get

$$\frac{1}{n} < \ln\left(1 + \frac{1}{n-1}\right) = \ln\frac{n}{n-1}$$

From this inequality and the second part of inequality (1) it follows that

$$\ln\frac{n+1}{n} < \frac{1}{n} < \ln\frac{n}{n-1}$$

Now, using inequalities (4), we write inequalities

$$\ln\frac{n+1}{n} < \frac{1}{n} < \ln\frac{n}{n-1},$$
$$\ln\frac{n+2}{n+1} < \frac{1}{n+1} < \ln\frac{n+1}{n},$$
$$\ln\frac{n+3}{n+2} < \frac{1}{n+2} < \ln\frac{n+2}{n+1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ln\frac{2n+1}{2n} < \frac{1}{2n} < \ln\frac{2n}{2n-1}$$

Adding then and taking into account that the sum of logarithms is equal to the logarithm of the product, we get

$$\ln\frac{(n+1)(n+2)(n+3)\ldots(2n+1)}{n(n+1)(n+2)\ldots 2n} < \frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n} <$$
$$< \ln\frac{n(n+1)(n+2)\ldots 2n}{(n-1)n(n+1)\ldots(2n-1)}$$

i. e.

$$\ln\frac{2n+1}{n} < \frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{1n} < \ln\frac{2n}{n-1} \tag{5}$$

Because

$\frac{2n+1}{n} = 2 + \frac{1}{n}$, that

$\lim\limits_{n\to\infty} \ln\frac{2n+1}{n} = \lim\limits_{n\to\infty} \ln\left(2 + \frac{1}{n}\right) = \ln 2$

Exactly the same from

$$\frac{2n}{n-1} = 2 + \frac{2}{n-1} \quad \text{that}$$

$$\lim_{n\to\infty} \ln \frac{2n}{n-1} = \ln 2$$

So, the extreme members of inequality (5) have the same limits. Consequently, the middle term has the same limit, i.e.

$$\lim_{n\to\infty}\left(\frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n}\right) = \lim_{n\to\infty} x_n = \ln 2$$

The task of 2. Putting $z_1 = 1,\ z_2 = 1 - \frac{1}{2},\ z_3 = 1 - \frac{1}{2} + \frac{1}{3}, ..., z_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} +$ $+ \frac{1}{5} - \frac{1}{6} + ... + (-1)^{n-1}\frac{1}{n},$ calculate $\lim_{n\to\infty} z_n$.

The solution. we have

$$z_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ... + \frac{1}{2n-1} - \frac{1}{2n} = \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + ... + \frac{1}{2n-1} + \frac{1}{2n}\right) -$$

$$2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + ... + \frac{1}{2n}\right) = \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + ... + \frac{1}{2n-1} + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right) =$$

$$\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}$$

In the previous task we put

$$x_n = \frac{1}{n} + \frac{1}{n+1} + ... + \frac{1}{2n}$$

Consequently, $z_{2n} = x_n - \frac{1}{n}$. But $\lim_{n\to\infty} x_n = \ln 2$. In this way,

$$\lim_{n\to\infty} z_{2n} = \lim_{n\to\infty}\left(z_n - \frac{1}{n}\right) = \ln 2$$

Note also, that $z_{2n+1} = z_{2n} + \frac{1}{2n+1}$, and hence $\lim_{n\to\infty} z_{2n+1} = \lim_{n\to\infty}\left(z_{2n} + \frac{1}{2n+1}\right) = \ln 2$

So, $\lim_{n\to\infty} z_n = \ln 2$.

The task of 3.

Series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n} + ...$$

is called harmonic series. Prove that harmonic series diverges.

The solution. according to inequality (1)

$$\frac{1}{n} > \ln \frac{n+1}{n}$$

Assuming $n = 1,2,3,...,n,$ we write n inequalities:

$$1 > \ln\frac{2}{1},$$
$$\frac{1}{2} > \ln\frac{3}{2},$$
$$\frac{1}{3} > \ln\frac{4}{3},$$
....
$$\frac{1}{n} > \ln\frac{n+1}{n}$$

Adding them, we get

$$x_n = 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n} > \ln\frac{2\cdot 3\cdot 4...(n+1)}{1\cdot 2\cdot 3...n} = \ln(n+1)$$

From this inequality it follows that $\lim_{n\to\infty} x_n \geq \lim_{n\to\infty}\ln(n+1) = \infty$

Therefore, the harmonic series diverges.

The task of 4. Find limit

$$\lim_{n\to\infty}\frac{1+2^{\alpha}+3^{\alpha}+...+n^{\alpha}}{n^{1+\alpha}}, \quad \alpha > 0$$

The solution. we first prove inequality

$$\frac{n^{\alpha+1}}{\alpha+1} < 1+2^{\alpha}+3^{\alpha}+...+n^{\alpha} < \frac{(n+1)^{\alpha+1}}{\alpha+1}, \quad \alpha > 0$$

Since $\alpha > 0$, then $\alpha+1>1$ and therefore

$$\left(1+\frac{1}{n}\right)^{1+\alpha} > 1+\frac{1+\alpha}{n},$$
$$\left(1-\frac{1}{n}\right)^{1+\alpha} > 1-\frac{1+\alpha}{n}$$

Multiplying these inequalities by

$n^{1+\alpha}$ we get

$$(n+1)^{1+\alpha} > n^{1+\alpha}+(1+\alpha)n^{\alpha},$$
$$(n+1)^{1+\alpha} > n^{1+\alpha}-(1+\alpha)n^{\alpha},$$

From these inequalities it follows that

$$\frac{n^{1+\alpha}-(n-1)^{1+\alpha}}{1+\alpha} < n^{\alpha} < \frac{(n+1)^{1+\alpha}-n^{1+\alpha}}{1+\alpha}.$$

We write these inequalities at values of $n = 1, 2, 3, ..., n$:

$$\frac{1}{1+\alpha} < 1 < \frac{2^{1+\alpha}-1}{1+\alpha},$$
$$\frac{2^{1+\alpha}-1}{1+\alpha} < 2^{\alpha} < \frac{3^{1+\alpha}-2^{1+\alpha}}{1+\alpha},$$
....
$$\frac{n^{1+\alpha}-(n-1)^{1+\alpha}}{1+\alpha} < n^{\alpha} < \frac{(n+1)^{1+\alpha}-n^{1+\alpha}}{1+\alpha}$$

Adding them we get

$$\frac{1}{1+\alpha} < 1+2^{\alpha}+3^{\alpha}+...+n^{\alpha} < \frac{(n+1)^{1+\alpha}}{1+\alpha} < \frac{(n+1)^{1+\alpha}}{1+\alpha}$$

From here

$$\frac{1}{1+\alpha} < \frac{1+2^{\alpha}+3^{\alpha}+...+n^{\alpha}}{n^{1+\alpha}} < \frac{\left(1+\frac{1}{n}\right)^{1+\alpha}}{1+\alpha}$$

The lest part of the last inequalities is the constant number $\frac{1}{1+\alpha}$, and the right side tends to the limit equal to $\frac{1}{1+\alpha}$, when *n* tends to infinity. Hence, the middle part of the inequalities tends to the same limit, i.e.

$$\lim_{n\to\infty} \frac{1+2^{\alpha}+3^{\alpha}+...+n^{\alpha}}{n^{1+\alpha}} = \frac{1}{1+\alpha}$$

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